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Threshold bounds for noisy bipartite states

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Abstract

For a nonseparable bipartite quantum state violating the Clauser–Horne–Shimony–Holt (CHSH) inequality, we evaluate amounts of noise breaking the quantum character of its statistical correlations under any generalized quantum measurements of Alice and Bob. Expressed in terms of the reduced states, these new threshold bounds can be easily calculated for any concrete bipartite state. A noisy bipartite state, satisfying the extended CHSH inequality and the perfect correlation form of the original Bell inequality for any quantum observables, neither necessarily admits a local hidden variable model nor exhibits the perfect correlation of outcomes whenever the same quantum observable is measured on both 'sides'.

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1. Introduction

The violation of Bell-type inequalities in the quantum case is used in many quantum information tasks. In reality, one, however, deals with noisy channels and, for a bipartite quantum state, it is important to estimate amounts of noise breaking the quantum character of its statistical correlations under quantum measurements of Alice and Bob.

In the present paper, we analyse this problem based our recent results in [1, 2] where, in a general setting, we introduced bipartite quantum states, *density source operator* (*DSO*) *states*, that satisfy the original CHSH inequality [3] under any generalized quantum measurements of Alice and Bob. A DSO state with the special dilation property, *a Bell class DSO state*, satisfies the perfect correlation form of the original Bell inequality [4] for any three quantum observables[1].

In section 2, we shortly list the main properties of DSO states specified in [1, 2] and further prove in a general setting that any DSO state satisfies a generalized version of the original CHSH inequality [3]—the extended CHSH inequality, which we introduced in [5].

[1] A classical state satisfies a CHSH-form inequality under any Alice and Bob classical measurements, ideal or randomized, and the perfect correlation form of the original Bell inequality for any three classical observables (see the appendix of [5] for a general proof of the latter).

In section 3, for an arbitrary nonseparable quantum state violating the original CHSH inequality [3], we introduce the amounts of noise sufficient for the resulting noisy state to represent a DSO state and a Bell class DSO state and, therefore, to satisfy the extended CHSH inequality [5] and the perfect correlation form of the original Bell inequality [4] for any quantum observables. A noisy bipartite state satisfying the extended CHSH inequality does not necessarily admit a local hidden variable (LHV) model[2] while a noisy bipartite state satisfying the perfect correlation form of the original Bell inequality does not necessarily exhibit the perfect correlation of outcomes if the same quantum observable is measured on both 'sides'[3].

In section 4, we specify our general bounds for some concrete nonseparable pure states and, in particular, demonstrate that, satisfying the perfect correlation form of the original Bell inequality for any three qubit observables, every separable noisy singlet does not exhibit the perfect correlation of outcomes whenever the same qubit observable is projectively measured on both 'sides'. If, in particular, the same spin observable is measured on both 'sides', the correlation function of a separable noisy singlet is negative. This explicitly points to the faulty character of the wide-spread opinion (expressed, for example, in [7]) that the validity of the perfect correlation form of the original Bell inequality is necessarily linked with the perfect correlation condition specified by Bell [4].

2. Bipartite quantum states exhibiting classical statistical correlations

According to our developments in [1, 2], for any state ρ on a separable complex Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, possibly infinite dimensional, there always exist self-adjoint trace class dilations[4] $T_\blacktriangleright$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ and $T_\blacktriangleleft$ on $\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$, not necessarily positive, defined by the relations[5]: $\mathrm{tr}^{(2)}_{\mathcal{H}_2}[T_\blacktriangleright] = \mathrm{tr}^{(3)}_{\mathcal{H}_2}[T_\blacktriangleright] = \rho$ and $\mathrm{tr}^{(1)}_{\mathcal{H}_1}[T_\blacktriangleleft] = \mathrm{tr}^{(2)}_{\mathcal{H}_1}[T_\blacktriangleleft] = \rho$. This definition implies: $\mathrm{tr}[T_\blacktriangleright] = \mathrm{tr}[T_\blacktriangleleft] = 1$.

We refer to any of these dilations as a *source operator* for a bipartite state ρ. Since any positive source operator is a density operator, we specify it as a *density source operator* (*DSO*).

If, for a bipartite state ρ, there exists a density source operator, we call this ρ a density source operator state or *a DSO state*, for short. The set of all DSO states on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is convex and includes all separable states and a variety of nonseparable states.

We say that a state ρ on $\mathcal{H} \otimes \mathcal{H}$ is *a DSO state of the Bell class* if ρ has a density source operator T with the special dilation property $\mathrm{tr}^{(1)}_{\mathcal{H}}[T] = \mathrm{tr}^{(2)}_{\mathcal{H}}[T] = \mathrm{tr}^{(3)}_{\mathcal{H}}[T] = \rho$. A Bell class DSO state may be separable or nonseparable. For example, as we proved in [1], every Werner state [6] on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 3$, separable or nonseparable, represents a Bell class DSO state. A two-qubit Werner state is a DSO state of the Bell class if it is separable.

The main properties of DSO states, important for applications, concern the classical character of their statistical correlations under quantum measurements of Alice and Bob with two measurement settings on each 'side'.

Consider a generalized Alice/Bob joint quantum measurement, with real-valued outcomes $\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2$ and performed upon a state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let this joint measurement be specified by a pair[6] (a, b) of measurement settings and described by positive operator-valued (*POV*) measures $M_1^{(a)}$ and $M_2^{(b)}$ of Alice and Bob, respectively. The joint probability

[2] In the sense formulated by Werner in [6].

[3] This correlation condition, sufficient for the derivation of the original Bell inequality, was introduced by Bell [4].

[4] The lower indices of $T_\blacktriangleright$, $T_\blacktriangleleft$ indicate the direction of extension.

[5] Here, $\mathrm{tr}^{(k)}_{\mathcal{H}_m}[\cdot]$ denotes the partial trace over the elements of $\mathcal{H}_m$ standing in the kth place of tensor products.

[6] For concreteness, the first argument in a pair refers to a marginal measurement (say of Alice) with outcomes λ_1 while the second argument refers to a Bob marginal measurement, with outcomes λ_2.

that outcomes λ_1 and λ_2 belong to subsets $B_1 \subseteq \Lambda_1$, $B_2 \subseteq \Lambda_2$, respectively, has the form[7]: $\mathrm{tr}\big[\rho\big(M_1^{(a)}(B_1) \otimes M_2^{(b)}(B_2)\big)\big]$. The expectation $\langle\lambda_1\lambda_2\rangle_\rho^{(a,b)}$ of the product $\lambda_1\lambda_2$ of the observed outcomes is given by

$$\langle\lambda_1\lambda_2\rangle_\rho^{(a,b)} := \int_{\Lambda_1\times\Lambda_2} \lambda_1\lambda_2 \,\mathrm{tr}\big[\rho\big(M_1^{(a)}(\mathrm{d}\lambda_1) \otimes M_2^{(b)}(\mathrm{d}\lambda_2)\big)\big] = \mathrm{tr}\big[\rho\big(W_1^{(a)} \otimes W_2^{(b)}\big)\big], \tag{1}$$

where $W_1^{(a)} := \int_{\Lambda_1} \lambda_1 M_1^{(a)}(\mathrm{d}\lambda_1)$ and $W_2^{(b)} := \int_{\Lambda_2} \lambda_2 M_2^{(b)}(\mathrm{d}\lambda_2)$ are bounded quantum observables on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

As we proved in a general setting in [1, 2]:

(i) a DSO state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfies the original CHSH inequality [3]:

$$\big|\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_1)} + \langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_2)} + \langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_1)} - \langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_2)}\big| \leqslant 2, \tag{2}$$

under any generalized quantum measurements of Alice and Bob with real-valued outcomes $|\lambda_1| \leqslant 1$, $|\lambda_2| \leqslant 1$ of an arbitrary spectral type;

(ii) a Bell class DSO state ρ on $\mathcal{H}\otimes\mathcal{H}$ satisfies the perfect correlation form of the original Bell inequality [4]:

$$\big|\langle\lambda_1\lambda_2\rangle_\rho^{(a,b_1)} - \langle\lambda_1\lambda_2\rangle_\rho^{(a,b_2)}\big| \leqslant 1 - \langle\lambda_1\lambda_2\rangle_\rho^{(b_1,b_2)}, \tag{3}$$

if, under generalized quantum measurements of Alice and Bob with real-valued outcomes $|\lambda_1| \leqslant 1$, $|\lambda_2| \leqslant 1$, the marginal POV measures obey the correlation condition

$$\int_{\Lambda_1} \lambda_1 M_1^{(b_1)}(\mathrm{d}\lambda_1) = \int_{\Lambda_2} \lambda_2 M_2^{(b_1)}(\mathrm{d}\lambda_2). \tag{4}$$

This operator relation does not imply the perfect correlation of outcomes, specified[8] by Bell in [4], and is always fulfilled in the case of Alice and Bob projective measurements of the same quantum observable on both 'sides'.

In view of (1), inequality (3) and condition (4) imply that a Bell class DSO state ρ on $\mathcal{H}\otimes\mathcal{H}$ satisfies[9] the perfect correlation form of the original Bell inequality

$$\begin{aligned} &|\mathrm{tr}[\rho(W_1\otimes W_2)] - \mathrm{tr}[\rho(W_1\otimes \widetilde{W}_2)]| \leqslant 1 - \mathrm{tr}[\rho(W_2\otimes\widetilde{W}_2)],\\ &|\mathrm{tr}[\rho(W_1\otimes W_2)] - \mathrm{tr}[\rho(\widetilde{W}_1\otimes W_2)]| \leqslant 1 - \mathrm{tr}[\rho(W_1\otimes\widetilde{W}_1)], \end{aligned} \tag{5}$$

for any bounded quantum observables $W_1, \widetilde{W}_1, W_2, \widetilde{W}_2$ on $\mathcal{H}$ with operator norms $\|\cdot\| \leqslant 1$. On the right-hand sides of inequalities (5), observables can be interchanged.

A general condition sufficient for an arbitrary DSO state to satisfy (5) is introduced in [2] (section 3, theorem 4).

In the appendix of this paper, extending further our above results [1, 2] on DSO states, we prove that a DSO state ρ on $\mathcal{H}_1\otimes\mathcal{H}_2$ satisfies a generalized version of (2)—the extended CHSH inequality:

$$\big|\gamma_{11}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_1)} + \gamma_{12}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_2)} + \gamma_{21}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_1)} + \gamma_{22}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_2)}\big| \leqslant 2\max_{i,k}|\gamma_{ik}|, \tag{6}$$

which we introduced in [5]. Here, γ_{ik} are any real coefficients obeying either of the relations:

$$\gamma_{11}\gamma_{12} = -\gamma_{21}\gamma_{22} \quad\text{or}\quad \gamma_{11}\gamma_{21} = -\gamma_{12}\gamma_{22} \quad\text{or}\quad \gamma_{11}\gamma_{22} = -\gamma_{12}\gamma_{21}. \tag{7}$$

Note that the extended CHSH inequality cannot be, in general, derived by rescaling[10] of the original CHSH inequality (2).

It should be stressed that a nonseparable DSO state does not necessarily admit an LHV model formulated in [6] while a Bell class DSO state, separable or nonseparable, does not necessarily exhibit Bell's perfect correlations (see footnote 8).

[7] See, for example, in [5] (section 3).
[8] For short, we further refer to the perfect correlation of outcomes if the same observable is measured on both 'sides' as *Bell's perfect correlations*.
[9] See [1] (theorem 2).
[10] Any rescaling of (2) results in a version of (6) with coefficients γ_{ik} satisfying only the last relation in (7).

3. Noisy bipartite quantum states

For an arbitrary state ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $\forall d_1, d_2 \geqslant 2$, violating the original CHSH inequality (2), let us now specify the amounts of noise sufficient for the noisy state $\eta_\rho(\beta) = \beta \frac{I_{\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}}}{d_1 d_2} + (1 - \beta)\rho$, $\beta \in (0, 1]$, to satisfy inequalities (2), (3) and (6).

Consider first such Alice and Bob generalized quantum measurements with real-valued outcomes $|\lambda_1| \leqslant 1, |\lambda_2| \leqslant 1$, where all product averages in the state $\sigma_{\text{noise}} = \frac{I_{\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}}}{d_1 d_2}$ are equal to zero: $\langle \lambda_1 \lambda_2 \rangle_{\sigma_{\text{noise}}}^{(a_i, b_k)} = 0, \forall i, k = 1, 2$. Under these joint measurements, the noisy state $\eta_\rho(\beta)$ satisfies the original CHSH inequality (2) whenever[11] $2\sqrt{2}(1 - \beta) \leqslant 2 \Leftrightarrow \beta \geqslant 1 - \frac{\sqrt{2}}{2}$ and the extended CHSH inequality (6) if $4(1 - \beta) \leqslant 2 \Leftrightarrow \beta \geqslant \frac{1}{2}$.

If, furthermore, $\langle \lambda_1 \lambda_2 \rangle_{\sigma_{\text{noise}}}^{(b_1, b_2)} = 0$, then the noisy state $\eta_\rho(\beta)$ satisfies both forms[12] of the original Bell inequality (3) whenever $2(1 - \beta) \leqslant 1 - (1 - \beta) \Leftrightarrow \beta \geqslant \frac{2}{3}$.

However, the above threshold bounds do not need to hold if, under Alice and Bob joint measurements, at least one of the product averages $\langle \lambda_1 \lambda_2 \rangle_{\sigma_{\text{noise}}}^{(a_i, b_k)} \neq 0$. Below, we specify the threshold bounds that are valid under any generalized quantum measurements of Alice and Bob.

Theorem 1. *Let a nonseparable state ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $d_1, d_2 \geqslant 2$, violate the original CHSH inequality (2). Denote by*[13]

$$\gamma_\rho := \min \left\{ d_1 \left\| \tau_\rho^{(1)} \right\|, d_2 \left\| \tau_\rho^{(2)} \right\| \right\} \geqslant 1 \tag{8}$$

the parameter characterizing ρ in terms of its reduced states $\tau_\rho^{(1)} := \mathrm{tr}_{\mathbb{C}^{d_2}}[\rho]$, $\tau_\rho^{(2)} := \mathrm{tr}_{\mathbb{C}^{d_1}}[\rho]$ on $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively. The noisy state

$$\eta_\rho(\beta) = \beta \frac{I_{\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}}}{d_1 d_2} + (1 - \beta)\rho, \quad \beta \in [\beta_{\text{CHSH}}(\rho), 1], \quad \beta_{\text{CHSH}}(\rho) := \frac{\gamma_\rho}{1 + \gamma_\rho}, \tag{9}$$

satisfies the extended CHSH inequality (6) under any generalized quantum measurements of Alice and Bob. In (9), $\beta_{\text{CHSH}}(\rho) \geqslant \frac{1}{2}, \forall \rho$.

Proof. Consider the decomposition $\rho = \sum \gamma_{nm, n_1 m_1} |e_n\rangle\langle e_{n_1}| \otimes |f_m\rangle\langle f_{m_1}|$ in orthonormal bases $\{e_n\}$ in $\mathbb{C}^{d_1}$ and $\{f_m\}$ in $\mathbb{C}^{d_2}$. For a noisy state $\eta_\rho(\beta)$, the operator

$$\begin{aligned} T_{\blacktriangleright}^{(\beta)} = (1 - \beta) \sum \gamma_{nm, n_1 m_1} |e_n\rangle\langle e_{n_1}| \otimes \{ |f_m\rangle\langle f_{m_1}| \otimes \xi_2 + \xi_2 \otimes |f_m\rangle\langle f_{m_1}| \} \\ - (1 - \beta)\tau_\rho^{(1)} \otimes \xi_2 \otimes \xi_2 + \beta \frac{I_{\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}}}{d_1 d_2^2} \end{aligned} \tag{10}$$

on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}$ represents a source operator. Here, ξ_2 is a density operator on $\mathbb{C}^{d_2}$.

If, in (10), the operator $Y^{(\beta)} := \beta \frac{I_{\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}}}{d_1 d_2^2} - (1 - \beta)\tau_\rho^{(1)} \otimes \xi_2 \otimes \xi_2$ is nonnegative, then the source-operator $T_{\blacktriangleright}^{(\beta)}$ is positive, that is, represents a density source operator. In view of the relation $-\|W\| I_{\mathcal{K}} \leqslant W \leqslant \|W\| I_{\mathcal{K}}$, valid for any bounded observable W on a Hilbert space $\mathcal{K}$, we derive

$$Y^{(\beta)} \geqslant \frac{\beta - d_1 d_2^2 \|\tau_\rho^{(1)}\| \|\xi_2\|^2 (1 - \beta)}{d_1 d_2^2} I_{\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}}. \tag{11}$$

[11] Here, we take into account that, in the quantum case, the maximal value of the left-hand side of (2) constitutes $2\sqrt{2}$ (the Cirel'son bound [8]).

[12] The perfect anticorrelation form of the original Bell inequality [4] corresponds to plus sign on the right-hand sides of (3), (5).

[13] $\|\cdot\|$ denotes the operator norm.

Therefore, $T_{\blacktriangleright}^{(\beta)}$ is a density source operator for any

$$\beta \geqslant \frac{d_1 d_2^2 \left\| \tau_\rho^{(1)} \right\| \|\xi_2\|^2}{1 + d_1 d_2^2 \left\| \tau_\rho^{(1)} \right\| \|\xi_2\|^2}. \tag{12}$$

Quite similarly, we construct the source operator $T_{\blacktriangleleft}^{(\beta)}$ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ and prove that $T_{\blacktriangleleft}^{(\beta)}$ is a density source operator for any $\beta \geqslant \frac{d_1^2 d_2 \|\tau_\rho^{(2)}\| \|\xi_1\|^2}{1+d_1^2 d_2 \|\tau_\rho^{(2)}\| \|\xi_1\|^2}$, where ξ_1 is a density operator on $\mathbb{C}^{d_1}$. Taking into account that, for $x \geqslant 0$, the function $\frac{x}{1+x}$ is monotone increasing and that, for a density operator ξ on $\mathbb{C}^d$, its operator norm $\frac{1}{d} \leqslant \|\xi\| \leqslant 1$, we choose in (11), (12) density operators ξ_1, ξ_2 with $\|\xi_1\| = \frac{1}{d_1}$, $\|\xi_2\| = \frac{1}{d_2}$.

Introducing further parameter (8) and noting that $\min\left\{\frac{d_1\|\tau_\rho^{(1)}\|}{1+d_1\|\tau_\rho^{(1)}\|}, \frac{d_2\|\tau_\rho^{(2)}\|}{1+d_2\|\tau_\rho^{(2)}\|}\right\} = \frac{\gamma_\rho}{1+\gamma_\rho}$, we derive that, for any $\beta \geqslant \frac{\gamma_\rho}{1+\gamma_\rho}$, the noisy state $\eta_\rho(\beta)$ is a DSO state and, therefore, satisfies (6). Since $d_1 \left\| \tau_\rho^{(1)} \right\| \geqslant 1$, $d_2 \left\| \tau_\rho^{(2)} \right\| \geqslant 1$, the parameter $\gamma_\rho \geqslant 1$ and, hence, $\frac{\gamma_\rho}{1+\gamma_\rho} \geqslant \frac{1}{2}$. □

Theorem 2. *Let a state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, separable or nonseparable, with equal reduced $\tau_\rho^{(1)} = \tau_\rho^{(2)} = \tau_\rho$, violate the perfect correlation form (5) of the original Bell inequality. The noisy state*

$$\eta_\rho(\beta) = \beta \frac{I_{\mathbb{C}^d \otimes \mathbb{C}^d}}{d^2} + (1-\beta)\rho, \qquad \beta \in [\beta_{\text{Bell}}(\rho), 1], \qquad \beta_{\text{Bell}}(\rho) := \frac{2\gamma_\rho^3}{1+2\gamma_\rho^3}, \tag{13}$$

(where $\gamma_\rho := d\|\tau_\rho\| \geqslant 1$) satisfies inequality (5) for any three quantum observables on $\mathbb{C}^d$ and inequality (3) under any generalized quantum measurements of Alice and Bob where the marginal POV measures obey the correlation condition (4). In (13), $\beta_{\text{Bell}}(\rho) \geqslant \frac{2}{3}, \forall \rho$.

Proof. Consider the decomposition $\rho = \sum \gamma_{nm,n_1m_1} |e_n\rangle\langle e_{n_1}| \otimes |e_m\rangle\langle e_{m_1}|$ in an orthonormal basis $\{e_n\}$ in $\mathbb{C}^d$. For a noisy state $\eta_\rho(\beta)$, the operator

$$\begin{aligned} T^{(\beta)} = (1-\beta) \sum \gamma_{nm,n_1m_1} \{ &|e_n\rangle\langle e_{n_1}| \otimes |e_m\rangle\langle e_{m_1}| \otimes \tau_\rho + |e_n\rangle\langle e_{n_1}| \otimes \tau_\rho \otimes |e_m\rangle\langle e_{m_1}| \\ &+ \tau_\rho \otimes |e_n\rangle\langle e_{n_1}| \otimes |e_m\rangle\langle e_{m_1}| \} - 2(1-\beta)\tau_\rho \otimes \tau_\rho \otimes \tau_\rho + \beta \frac{I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d}}{d^3} \end{aligned} \tag{14}$$

on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ represents a source operator with the special dilation property

$$\mathrm{tr}_{\mathbb{C}^d}^{(k)}[T^{(\beta)}] = \eta_\rho(\beta), \qquad k = 1, 2, 3. \tag{15}$$

If, on the left-hand side of (14), the operator $\widetilde{Y}^{(\beta)} := \beta \frac{I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d}}{d^3} - 2(1-\beta)\tau_\rho \otimes \tau_\rho \otimes \tau_\rho$ is nonnegative, then the source-operator $T^{(\beta)}$ is positive, that is, represents a density source operator. Evaluating $\widetilde{Y}^{(\beta)}$ quite similarly as in (11), we derive $\widetilde{Y}^{(\beta)} \geqslant \frac{\beta - 2d^3\|\tau_\rho\|^3(1-\beta)}{d^3} I_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d}$. Therefore, for any $\beta \geqslant \frac{2\gamma_\rho^3}{1+2\gamma_\rho^3}$, the source operator $T^{(\beta)}$, with the special dilation property (15), is a density source operator. This means that the noisy state $\eta_\rho(\beta)$, $\beta \in \left[\frac{2\gamma_\rho^3}{1+2\gamma_\rho^3}, 1\right]$, is a Bell class DSO state and, therefore, satisfies inequality (5) for any three quantum observables on $\mathbb{C}^d$ and inequality (3) under condition (4). We have $\frac{2\gamma_\rho^3}{1+2\gamma_\rho^3} \geqslant \frac{\gamma_\rho}{1+\gamma_\rho}$ and $\frac{2\gamma_\rho^3}{1+2\gamma_\rho^3} \geqslant \frac{2}{3}$. □

Note that if ρ is an arbitrary state on $\mathcal{H} \otimes \mathcal{H}$ satisfying the original Bell inequality (5) for any bounded quantum observables on $\mathcal{H}$, then this ρ does not violate the extended CHSH inequality (6) and, in particular, the original CHSH inequality (2). Hence, for a state ρ on $\mathcal{H} \otimes \mathcal{H}$, the validity of the original CHSH inequality (2) is necessary for the validity of the original Bell inequality (5). Theorems 1 and 2 imply.

Corollary 1. *Let a nonseparable state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, with equal reduced $\tau_\rho^{(1)} = \tau_\rho^{(2)} = \tau_\rho$, violate the original CHSH inequality (2). The noisy state $\beta \frac{I_{\mathbb{C}^d \otimes \mathbb{C}^d}}{d^2} + (1-\beta)\rho$, $\beta \in (0, 1]$, does not violate the extended CHSH inequality (6) if $\beta \geqslant \beta_{\text{CHSH}}(\rho) := \frac{\gamma_\rho}{1+\gamma_\rho}$ and does not violate the perfect correlation form (5) of the Bell inequality whenever $\beta \geqslant \beta_{\text{Bell}}(\rho) := \frac{2\gamma_\rho^3}{1+2\gamma_\rho^3}$. We have $\beta_{\text{CHSH}}(\rho) \geqslant \frac{1}{2}$, $\beta_{\text{Bell}}(\rho) \geqslant \frac{2}{3}$ and $\beta_{\text{Bell}}(\rho) > \beta_{\text{CHSH}}(\rho)$.*

The threshold bounds specified in theorems 1, 2 and corollary 1 are sufficient. Therefore, for a nonseparable bipartite state, the threshold amounts of noise β_{CHSH}, β_{Bell}, defined by (9) and (13), are not necessarily least.

4. Examples

Consider on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, the nonseparable pure state

$$\rho_{d,\vartheta} = |\phi_{d,\vartheta}\rangle\langle\phi_{d,\vartheta}|, \qquad \phi_{d,\vartheta} = \frac{1}{\sqrt{d}} \sum_{n=1}^{d} \exp(\mathrm{i}\vartheta_n) e_n \otimes e_n, \tag{16}$$

specified by a sequence $\vartheta = \{\vartheta_n\}$ of real phases. This state violates[14] the original CHSH inequality (2) and, therefore, the original Bell inequality (5).

For the nonseparable state $\rho_{d,\vartheta}$, both reduced states are equal to $\frac{I_{\mathbb{C}^d}}{d}$, the parameter $\gamma_{\rho_{d,\vartheta}} = 1$ and, due to corollary 1, the threshold amounts of noise constitute $\beta_{\text{CHSH}}(\rho_{d,\vartheta}) = \frac{1}{2}$, $\beta_{\text{Bell}}(\rho_{d,\vartheta}) = \frac{2}{3}$.

In view of the Peres separability criterion [9], for any $\beta \in \left[0, \frac{d}{d+1}\right)$, the state $\eta_{\rho_{d,\vartheta}}(\beta) = \beta \frac{I_{\mathbb{C}^d \otimes \mathbb{C}^d}}{d^2} + (1-\beta)\rho_{d,\vartheta}$ is nonseparable[15]. Hence, for any separable noisy state $\eta_{\rho_{d,\vartheta}}(\beta)$, the value of β cannot be less than $\frac{d}{d+1}$. Since $\beta_{\text{Bell}}(\rho_{d,\vartheta}) = \frac{2}{3}$ and $\frac{d}{d+1} \geqslant \frac{2}{3}$, $\forall d \geqslant 2$, we conclude that *every separable admixture of $\rho_{d,\vartheta}$ to $\frac{I_{\mathbb{C}^d \otimes \mathbb{C}^d}}{d^2}$ does not violate the perfect correlation form* (5) *of the original Bell inequality.*

The latter property holds also for any separable noisy Bell state on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Recall that the Bell states have the form:

$$\phi^{(\pm)} = \tfrac{1}{\sqrt{2}}(e_1 \otimes e_1 \pm e_2 \otimes e_2), \qquad \psi^{(\pm)} = \tfrac{1}{\sqrt{2}}(e_1 \otimes e_2 \pm e_2 \otimes e_1), \tag{17}$$

and that a noisy Bell state is separable iff $\beta \geqslant \frac{2}{3}$. For any of the Bell states (17), the reduced states are given by $\frac{I_{\mathbb{C}^2}}{2}$, parameter (8) is equal to 1 and, therefore, due to corollary 1, the threshold amounts of noise $\beta_{\text{CHSH}} = \frac{1}{2}$, $\beta_{\text{Bell}} = \frac{2}{3}$.

Thus, *every separable admixture of a Bell state to $\frac{I_{\mathbb{C}^2 \otimes \mathbb{C}^2}}{4}$ does not violate the perfect correlation form (5) of the original Bell inequality.*

Note that the noisy singlet $\eta_{\psi^{(-)}}\left(\frac{1}{2}\right) = \frac{I_{\mathbb{C}^2 \otimes \mathbb{C}^2}}{8} + \frac{|\psi^{(-)}\rangle\langle\psi^{(-)}|}{2}$, corresponding to the threshold value $\beta_{\text{CHSH}} = \frac{1}{2}$, represents the two-qubit nonseparable Werner state [6], specified in [6] by the parameter $\Phi = -\frac{1}{4}$.

Let us now explicitly demonstrate that, satisfying the perfect correlation form (5) of the original Bell inequality for any three qubit observables (with operator norms $\|\cdot\| \leqslant 1$), the separable noisy singlet $\eta_{\psi^{(-)}}(\beta) = \beta \frac{I^2_{\mathbb{C}} \otimes \mathbb{C}^2}{4} + (1-\beta)|\psi^{(-)}\rangle\langle\psi^{(-)}|$, $\beta \in \left[\frac{2}{3}, 1\right)$, does not exhibit Bell's perfect correlations (see footnote 8).

[14] If all $\vartheta_n = 0$, then this state violates (2) maximally.
[15] The partial transpose of $\eta_{\rho_{d,\vartheta}}(\beta)$ has the eigenvalue $\frac{\beta(d+1)-d}{d^2}$, which is negative for any $\beta < \frac{d}{d+1}$.

Recall that, in an orthonormal basis $\{e_k, k = 1, 2\}$ in $\mathbb{C}^2$, a generic qubit observable has the form $W_{\alpha,n} = \alpha I_{\mathbb{C}^2} + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$, where (i) α is any real number; (ii) $\sigma_x := |e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|, \sigma_y := i(|e_2\rangle\langle e_1| - |e_1\rangle\langle e_2|), \sigma_z := |e_1\rangle\langle e_1| - |e_2\rangle\langle e_2|$ are self-adjoint operators, represented in a basis $\{e_k\}$ by the Pauli matrices; (iii) $n = (n_x, n_y, n_z)$ is a vector in $\mathbb{R}^3$. The eigenvalues of $W_{\alpha,n}$ are equal to $\alpha \pm |n|$, where $|n| := \sqrt{n_x^2 + n_y^2 + n_z^2}$.

For the noisy singlet $\eta_{\psi^{(-)}}(\beta)$, the correlation function

$$\mathrm{tr}[\eta_{\psi^{(-)}}(\beta)(W_{\alpha,n} \otimes W_{\alpha,n})] = \alpha^2 - |n|^2(1 - \beta) \tag{18}$$

is negative[16] for any qubit observable $W_{\alpha,n}$ with $\alpha^2 < |n|^2(1 - \beta)$, in particular, for any spin observable[17].

Furthermore, in the case of Alice and Bob projective measurements of the same qubit observable $W_{\alpha,n}$, where $|n| \neq 0$, in the noisy singlet $\eta_{\psi^{(-)}}(\beta)$ the joint probabilities constitute:

$$\begin{aligned} &\mathrm{Prob}\{\lambda_1 = \alpha \pm |n|, \lambda_2 = \alpha \pm |n|\} = \frac{\beta}{4}, \\ &\mathrm{Prob}\{\lambda_1 = \alpha \pm |n|, \lambda_2 = \alpha \mp |n|\} = \frac{1}{2} - \frac{\beta}{4}. \end{aligned} \tag{19}$$

Hence, given an outcome, say of Bob, the conditional probability that Alice observes a different outcome is equal to $1 - \frac{\beta}{2}$ while the conditional probability that Alice observes the same outcome is $\frac{\beta}{2}$.

Thus, satisfying the perfect correlation form (5) of the original Bell inequality under Alice and Bob projective measurements of any three qubit observables, the separable noisy singlet $\eta_{\psi^{(-)}}(\beta), \beta \in \left[\frac{2}{3}, 1\right)$, *does not* exhibit the perfect correlation of outcomes whenever the same qubit observable is measured on both 'sides'.

Appendix

Below, in theorem 3, we exploit the following property of probability measures on product spaces.

Let π be a probability distribution with outcomes in $\Lambda_1 \times \Lambda_2 \times \Lambda_2$. For any subsets[18] $B_1 \subseteq \Lambda_1, B' \subseteq \Lambda_2 \times \Lambda_2$, the relation $\pi(\Lambda_1 \times B') = 0$ implies $\pi(B_1 \times B') = 0$. Hence, for any $B_1 \subseteq \Lambda_1$, the probability distribution $\pi(B_1 \times \cdot)$ of outcomes in $\Lambda_2 \times \Lambda_2$ is absolutely continuous[19] with respect to the marginal probability distribution $\pi(\Lambda_1 \times \cdot)$.

The latter implies that, for any subsets $B_1 \subseteq \Lambda_1, B_2, \widetilde{B}_2 \subseteq \Lambda_2$, the probability distribution $\pi(B_1 \times B_2 \times \widetilde{B}_2)$ admits the representation

$$\pi(B_1 \times B_2 \times \widetilde{B}_2) = \int_{B_2 \times \widetilde{B}_2} \nu(B_1|\lambda_2, \widetilde{\lambda}_2)\pi(\Lambda_1 \times \mathrm{d}\lambda_2 \times \mathrm{d}\widetilde{\lambda}_2), \tag{A.1}$$

where (i) for π-almost all $\lambda_2, \widetilde{\lambda}_2 \in \Lambda_2$, the mapping $\nu(\cdot|\lambda_2, \widetilde{\lambda}_2)$ is a probability distribution of outcomes in Λ_1; (ii) for any subset $B_1 \subseteq \Lambda_1$, the real-valued function $\nu(B_1|\cdot, \cdot) : \Lambda_2 \times \Lambda_2 \to [0, 1]$ is measurable.

Theorem 3. *A DSO state ρ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfies the extended CHSH inequality (6) under any generalized quantum measurements of Alice and Bob, with real-valued outcomes $|\lambda_1| \leqslant 1, |\lambda_2| \leqslant 1$ of an arbitrary spectral type.*

[16] Negativity of the correlation function (18) rules out Bell's perfect correlations.
[17] For a spin observable, $\alpha = 0$, $|n| = 1$.
[18] For simplicity, we do not specify here the notion of a σ-algebra of subsets of Λ.
[19] On this notion, see, for example, [10].

Proof. Let a DSO state ρ have a density source operator $T_\blacktriangleright$ on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$. Due to the definition (see section 2) of a DSO $T_\blacktriangleright$, we have

$$\begin{aligned}
&\mathrm{tr}\big[\rho\big(M_1^{(a)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2)\big)\big] = \mathrm{tr}\big[T_\blacktriangleright\big(M_1^{(a)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes I_{\mathcal{H}_2}\big)\big],\\
&\mathrm{tr}\big[\rho\big(M_1^{(a)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_2)}(\mathrm{d}\tilde{\lambda}_2)\big)\big] = \mathrm{tr}\big[T_\blacktriangleright\big(M_1^{(a)}(\mathrm{d}\lambda_1) \otimes I_{\mathcal{H}_2} \otimes M_2^{(b_2)}(\mathrm{d}\tilde{\lambda}_2)\big)\big].
\end{aligned} \tag{A.2}$$

Due to the normalization of a POV measure, the probability distributions, standing on the right-hand sides of (A2), constitute the marginals of the probability distribution $\mathrm{tr}\big[T_\blacktriangleright\big(M_1^{(a)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(\mathrm{d}\tilde{\lambda}_2)\big)\big]$. The latter and the representation (1) imply

$$\begin{aligned}
&\gamma_{11}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_1)} + \gamma_{12}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_2)} + \gamma_{21}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_1)} + \gamma_{22}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_2)}\\
&= \int_{\Lambda_1\times\Lambda_2\times\Lambda_2} (\gamma_{11}\lambda_1\lambda_2 + \gamma_{12}\lambda_1\widetilde{\lambda}_2)\, \mathrm{tr}\big[T_\blacktriangleright\big(M_1^{(a_1)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(\mathrm{d}\widetilde{\lambda}_2)\big)\big]\\
&+ \int_{\Lambda_1\times\Lambda_2\times\Lambda_2} (\gamma_{21}\widetilde{\lambda}_1\lambda_2 + \gamma_{22}\widetilde{\lambda}_1\widetilde{\lambda}_2)\, \mathrm{tr}\big[T_\blacktriangleright\big(M_1^{(a_2)}(\mathrm{d}\widetilde{\lambda}_1) \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(\mathrm{d}\widetilde{\lambda}_2)\big)\big],
\end{aligned} \tag{A.3}$$

where $\Lambda_1, \Lambda_2 \subseteq [-1, 1]$. Due to (A1), we have

$$\begin{aligned}
&\mathrm{tr}\big[T_\blacktriangleright\big(M_1^{(a_1)}(\mathrm{d}\lambda_1) \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(\mathrm{d}\widetilde{\lambda}_2)\big)\big]\\
&\qquad = \nu_{a_1,b_1,b_2}(\mathrm{d}\lambda_1|\lambda_2, \widetilde{\lambda}_2)\, \mathrm{tr}\big[T_\blacktriangleright\big(I_{\mathcal{H}_1} \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(\mathrm{d}\widetilde{\lambda}_2)\big)\big],\\
&\mathrm{tr}\big[T_\blacktriangleright\big(M_1^{(a_2)}(\mathrm{d}\widetilde{\lambda}_1) \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(d\widetilde{\lambda}_2)\big)\big]\\
&\qquad = \nu_{a_2,b_1,b_2}(\mathrm{d}\widetilde{\lambda}_1|\lambda_2, \widetilde{\lambda}_2)\, \mathrm{tr}\big[T_\blacktriangleright\big(I_{\mathcal{H}_1} \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(\mathrm{d}\widetilde{\lambda}_2)\big)\big],
\end{aligned} \tag{A.4}$$

where $\nu_{a_1,b_1,b_2}(\cdot|\lambda_2, \widetilde{\lambda}_2)$ and $\nu_{a_2,b_1,b_2}(\cdot|\lambda_2, \widetilde{\lambda}_2)$ are probability distributions of outcomes in Λ_1. Introducing the probability distribution

$$\begin{aligned}
&\mu(\mathrm{d}\lambda_1 \times \mathrm{d}\widetilde{\lambda}_1 \times \mathrm{d}\lambda_2 \times \mathrm{d}\widetilde{\lambda}_2)\\
&= \nu_{a_1,b_1,b_2}(\mathrm{d}\lambda_1|\lambda_2, \widetilde{\lambda}_2)\nu_{a_2,b_1,b_2}(\mathrm{d}\widetilde{\lambda}_1|\lambda_2, \widetilde{\lambda}_2)\, \mathrm{tr}\big[T_\blacktriangleright\big(I_{\mathcal{H}_1} \otimes M_2^{(b_1)}(\mathrm{d}\lambda_2) \otimes M_2^{(b_2)}(d\widetilde{\lambda}_2)\big)\big],
\end{aligned} \tag{A.5}$$

we rewrite (A3) in the form

$$\begin{aligned}
&\gamma_{11}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_1)} + \gamma_{12}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_2)} + \gamma_{21}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_1)} + \gamma_{22}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_2)}\\
&= \int_{\Lambda_1\times\Lambda_1\times\Lambda_2\times\Lambda_2} (\gamma_{11}\lambda_1\lambda_2 + \gamma_{12}\lambda_1\widetilde{\lambda}_2 + \gamma_{21}\widetilde{\lambda}_1\lambda_2 + \gamma_{22}\widetilde{\lambda}_1\widetilde{\lambda}_2)\mu(\mathrm{d}\lambda_1 \times \mathrm{d}\widetilde{\lambda}_1 \times \mathrm{d}\lambda_2 \times d\widetilde{\lambda}_2).
\end{aligned} \tag{A.6}$$

Taking into account that, for any real numbers $|\lambda_1| \leqslant 1$, $|\lambda_2| \leqslant 1$, the inequality

$$\big|\gamma_{11}\lambda_1\lambda_2 + \gamma_{12}\lambda_1\widetilde{\lambda}_2 + \gamma_{21}\widetilde{\lambda}_1\lambda_2 + \gamma_{22}\widetilde{\lambda}_1\widetilde{\lambda}_2\big| \leqslant 2\max_{i,k}|\gamma_{ik}| \tag{A.7}$$

holds with any real coefficients γ_{ik}, satisfying either of the relations in (7), we derive

$$\begin{aligned}
&\big|\gamma_{11}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_1)} + \gamma_{12}\langle\lambda_1\lambda_2\rangle_\rho^{(a_1,b_2)} + \gamma_{21}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_1)} + \gamma_{22}\langle\lambda_1\lambda_2\rangle_\rho^{(a_2,b_2)}\big|\\
&\leqslant \int_{\Lambda_1\times\Lambda_1\times\Lambda_2\times\Lambda_2} |\gamma_{11}\lambda_1\lambda_2 + \gamma_{12}\lambda_1\widetilde{\lambda}_2 + \gamma_{21}\widetilde{\lambda}_1\lambda_2 + \gamma_{22}\widetilde{\lambda}_1\widetilde{\lambda}_2|\mu(\mathrm{d}\lambda_1 \times \mathrm{d}\widetilde{\lambda}_1 \times \mathrm{d}\lambda_2 \times \mathrm{d}\widetilde{\lambda}_2)\\
&\leqslant 2\max_{i,k}|\gamma_{ik}|.
\end{aligned} \tag{A.8}$$

The validity of (6) for a DSO state ρ that has a density source-operator $T_\blacktriangleleft$ is proved quite similarly. □

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